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Engineering Notes

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Output Feedback Control of Two-Dimensional Aeroelastic System

Péter Baranyi*
Hungarian Academy of Sciences,
H-1111 Budapest, Hungary

I. Introduction

TENSOR Product (TP) model transformation was used in recent publications to show that linear-parameter-varying (LPV) models can be represented by the parameter-varying convex combination of linear time-invariant systems (LTI), termed TP models, whereupon linear matrix inequality (LMI) techniques in the parallel distributed compensation (PDC) design framework can immediately be executed to define controllers according to various different control specifications.[1] Based on this control design approach, Ref. 1 presents a state-variable feedback control design of the prototypical aeroelastic wing section and also claims that the TP model of the prototypical aeroelastic wing section is a gateway to LMI-based designs of the prototypical aeroelastic wing section.

The main objective of this Note is to derive an observer for the prototypical aeroelastic wing section via LMI-based design to estimate the practically unmeasurable state values from the output values and design an output feedback control strategy using the previously published state-feedback controller in Ref. 1.

The TP model transformation results in a tight convex hull of the LTI systems. The main novelty of this Note from a theoretical point of view is that in many cases the tight convex hull of the LTI systems does not lead to feasible LMIs in the case of observer design. This simply means that the TP model transformation, in its published form, is not applicable to observer design. In this Note we propose a method for extending the TP model transformation to yield a different kinds of convex hull of the LTI systems, which leads to feasible LMIs in the observer design process. The stability of the resulting output feedback control systems is guaranteed by the LMIs. This paper also presents numerical simulations to show the effectiveness of the resulting observer and the output feedback control design strategy.

This Note uses the nomenclature and the basic definitions (linear parameter-varying model, TP model transformation) of Ref. 1. Detailed description of the LPV model of the prototypical aeroelastic wing section is also omitted here; for this also see Ref. 1.

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*Head, Cognitive Vision Research, Computer and Automation Research Institute, Kende u. 13-17; also Integrated Intelligent Systems Japanese–Hungarian Laboratory, Műegyetem rakpart 3, H-1111 Budapest, Hungary; baranyi@tmit.bme.hu.

II. Extending the Tensor Product Model Transformation with Inverse and Relaxed Normality

This section is one of the main novelties of the Note. The TP model transformation is detailed in Ref. 1. The second step of the TP model transformation applies SN (sum-normalization), NN (nonnegativeness), and NO (normality) transformations. The SN and NN transformations are used to ensure convex combination; namely, they guarantee Eq. (11) of Ref. 1. The convex combination is required by the PDC framework. A number of examples show that the tight convex hull of the LTI systems relaxes the LMIs in the PDC controller design. Therefore the NO transformation was proposed with the SN and NN transformations in the TP model transformation. We experienced, however, that the observer design is relaxed (leads to relaxed LMIs) if we apply inverse NO (INO), and simultaneously relaxed NO (RNO) transformations. This condition has been defined by Ref. 2 in a different context together with a construction algorithm. Here, we present a new, more effective algorithm for INO and RNO transformations.

We do not detail the TP model transformation and transformations SN, NN, and NO here; we refer to Ref. 1 instead. We show how to execute INO and RNO transformations instead of the NO transformation in the singular value decomposition, which is the core of the second step of the TP model transformation.

First of all let us define the SN, INO, and RNO coefficient functions.

Definition 1: The vector $\boldsymbol{w}(p)$ of coefficient functions $w_n(p)$, $n = 1, \ldots, N$, is SN (sum normalization) if $\forall p : \mathrm{sum}_n(w_n(p_n) = 1)$.

Definition 2: The vector $\boldsymbol{w}(p)$ of coefficient functions $w_n(p)$, $n = 1, \ldots, N$, is INO (inverse NO) if $\forall n : \min_{p_n}(w_n(p_n) = 0)$.

Definition 3: The vector $\boldsymbol{w}(p)$ of coefficient functions $w_n(p)$, $n = 1, \ldots, N$, is RNO (relaxed NO) if $\forall n : \max_{p_n}(w_n(p_n) = a)$. Namely, the maxima of the coefficient functions are equal. If vector $\boldsymbol{w}(p)$ is SN and INO then obviously $0 \le a \le 1$.

Let $\boldsymbol{M}$ be an $n \times m$ matrix of rank r_0, and let the singular value decomposition (SVD) of $\boldsymbol{M}$ be

$$\boldsymbol{M} = \boldsymbol{U}^{(1)}\boldsymbol{D}^{(1)}\boldsymbol{V}^{(1)T} \tag{1}$$

where the sizes of $\boldsymbol{U}^{(1)}, \boldsymbol{D}^{(1)}$, and $\boldsymbol{V}^{(1)}$ are $n \times r_0$, $r_0 \times r_0$, and $m \times r_0$, respectively. In the following, we show a sequence of procedures that yields another decomposition of $\boldsymbol{M}$,

$$\boldsymbol{M} = \boldsymbol{UDV}^T \tag{2}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are SN, INO, and RNO and they have r_0 or $r_0 + 1$ columns.

An SN-type matrix decomposition algorithm is presented in Ref. 2 to construct the form

$$\boldsymbol{U}^{(1)} = \boldsymbol{U}^{(2)} \cdot \Theta^{(1)} \tag{3}$$

where $\boldsymbol{U}^{(2)}$ is SN and it is of size $n \times r_0$ or $n \times (r_0 + 1)$. Here we present a new algorithm providing

$$\boldsymbol{U}^{(2)} = \boldsymbol{U} \cdot \Theta^{(2)} \tag{4}$$

where $\boldsymbol{U}$ is SN, INO, and RNO and it is of the same size as $\boldsymbol{U}^{(2)}$. Equations (3) and (4) together with analogous decomposition of $\boldsymbol{V}^{(1)}$, can be substituted into Eq. (1) to get the required form (2).

III. Involving the Relaxed and Inverse Normality Properties

A. A Geometrical Interpretation of the Problem

In what follows, we describe an invertible, inhomogeneous linear transformation that transforms the row vectors of $\boldsymbol{U}^{(2)}$ to the rows of an SN, INO, and RNO matrix $\boldsymbol{U}$. The inverse of this transformation, applied on an $r \times r$ identity matrix, will provide Θ in Eq. (4). To make our steps more illustrative, we use a geometrical interpretation of Eq. (4), proposed in Ref. 2.

The rows of $\boldsymbol{U}^{(2)}$ and $\Theta^{(2)}$ are coordinates of r-dimensional points. Note the following geometrical considerations:

1) $\boldsymbol{U}^{(2)}$ and $\boldsymbol{U}$ are SN and $\Theta^{(2)}$ is also SN in Eq. (4). The fact that $\boldsymbol{U}^{(2)}$ and $\Theta^{(2)}$ are SN, means that the points determined by their rows lie in an $(r-1)$-dimensional hyperplane of the r-dimensional space. We can project all of the points into the $(r-1)$-dimensional space by erasing the last column of $\boldsymbol{U}^{(2)}$ and $\Theta^{(2)}$, yielding $\bar{\boldsymbol{U}}^{(2)}$ and $\bar{\Theta}^{(2)}$. The $(r-1)$-dimensional points determined by these truncated matrices are U_i and T_i, respectively.

2) The INO property of $\boldsymbol{U}$ means that all U_i points are inside the $(r-1)$-dimensional simplex determined by the points T_i and each face of the simplex contains at least one of the U_i points.

B. One Step of the Algorithm

If $r = 2$, the RNO and INO decomposition is trivial. Here we discuss the $r = 3$ case, which is also needed in wing-section observer design. Let a be a nonnegative scalar and let $A_{i,j}$ denote the (i, j)th element of an arbitrary matrix $\boldsymbol{A}$. We regard the following transformations transforming the row vectors of $\bar{\boldsymbol{U}}^{(2)}$ to $\bar{\boldsymbol{U}}^{D}(a)$ (see also Fig. 1):

$$A : \bar{U}^{A}_{i,j}(a) = \bar{U}^{(2)}_{i,j} + \frac{a-1}{r-1}\left(\sum_{k=1}^{r-1} \bar{U}_{i,k} - 1\right)$$

$$B : \bar{U}^{B}_{i,j}(a) = \bar{U}^{A}_{i,j}(a) - \min_{l}\left(\bar{U}^{A}_{l,j}(a)\right)$$

$$C : \bar{U}^{C}_{i,j}(a) = \frac{\bar{U}^{B}_{i,j}(a)}{\max_l\left(\bar{U}^{B}_{l,j}(a)\right)}$$

$$D : \bar{U}^{D}_{i,j}(a) = \frac{\bar{U}^{C}_{i,j}(a)}{\max_l\left(\sum_{k=1}^{r-1} \bar{U}^{C}_{l,k}(a)\right)} \tag{5}$$

We add an rth column to the $n \times (r-1)$ matrix $\bar{\boldsymbol{U}}^{D}(a)$ to make it SN. The resulting matrix $\boldsymbol{U}^{D}(a)$ is INO (the minima of the first $r-1$ columns are 0 because (5)/B and the minimum of the last one is 0 because (5)/D), and it is almost RNO: the maxima of the 1., 2., …, $r-1$th columns are equal (which follows from the transformation (5)/C). The maximum of the rth column depends on a. The RNO condition is satisfied if the maxima of the rth and the first column are equal, that is, if their difference $f(a)$ is zero:

$$f(a) = 1 - \min_{l}\left(\sum_{k=1}^{r-1} U^{D}_{l,k}(a)\right) - \max_{m}\left(U^{D}_{m,l}(a)\right) = 0 \tag{6}$$

Notice that

$$\lim_{a\to\infty} f(a) = 1 - 0 - \frac{1}{r-1} > 0 \tag{7}$$

$$f(0) = 1 - 1 - \max_{m}\left(U^{D}_{m,l}(0)\right) < 0 \tag{8}$$

(see also Fig. 2). Thus, Eq. (6) has a positive solution a because the function $f(a)$ is continuous. We can solve Eq. (6) numerically and produce the matrix $\boldsymbol{U} = \bar{\boldsymbol{U}}^{D}(a)$ satisfying the SN, INO and RNO conditions.

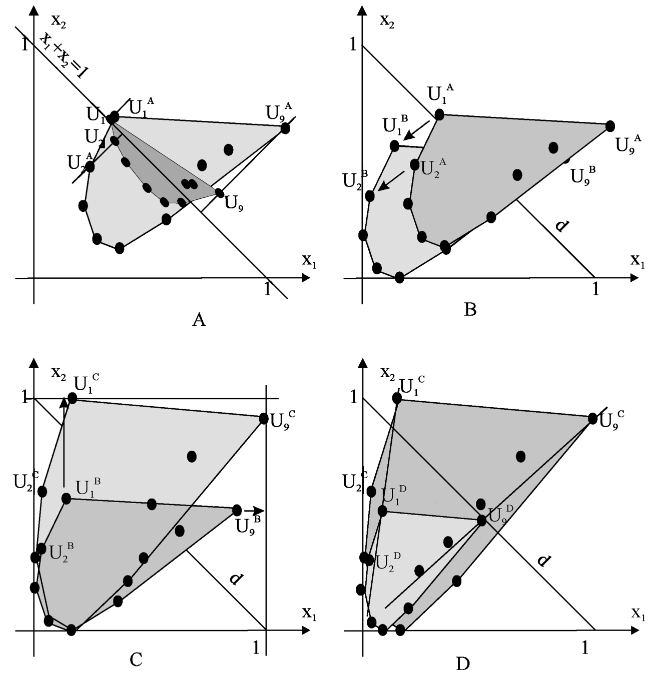

Fig. 1 The four steps of the transformation (5) illustrated by an example with $r = 3$ and $n = 9$. A) Perpendicular stretching by a from the line d ($a = 4$ in the figure), which would mean projection to d if $a = 0$. B) Shifting to the axes. C) Perpendicular stretching from the two coordinate axes to fill the unit square. D) Enlarging from the origin to hit line d.

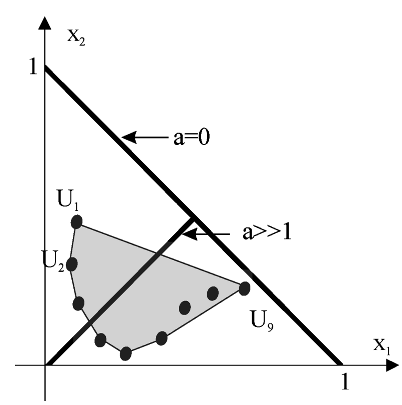

Fig. 2 An example of the transformation (5) at $r = 3$, if $a = 0$ and if $a \gg 1$.

We remark that equation (8) does not necessarily hold if $r > 3$, but the presented algorithm can be improved to be applicable in this more general case; details are omitted here.

IV. Output Feedback Design Control

A. Tenson Product Model Representations of the Prototypical Aeroelastic Wing Section

This section presents two TP model representations of the prototypical aeroelastic wing section. The first one is for the controller design. The second one is for the observer design.

TP Model 1: TP model 1 is derived in Ref. 1. The resulting coefficient functions are depicted in Fig. 3 of Ref. 1. The resulting $3 \times 2 = 6$ LTI systems are also given in Ref. 1.

TP Model 2: For observer design the present work executes the TP model transformation with the proposed INO and RNO transformations over the same $M_1 \times M_2$ hyper grid and in the same Ω as in the case of TP model 1. The resulting coefficient functions are

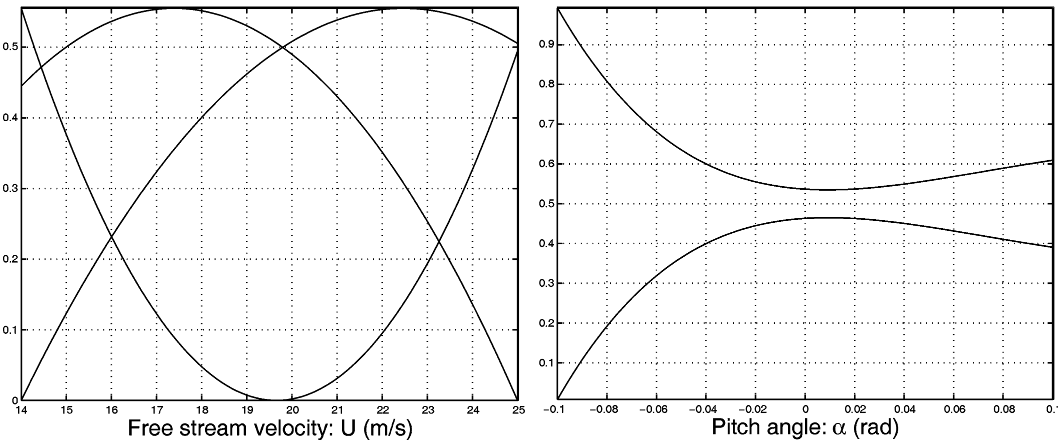

Fig. 3 Coefficient functions of TP model 2 on the dimensions α and U.

depicted in Fig. 3. The resulting $3 \times 2 = 6$ LTI systems are

$$\boldsymbol{A}_{1,1} = 10^3 \begin{pmatrix} 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 \\ -0.2314 & 0.0127 & -0.0029 & -0.0001 \\ 0.2780 & -1.2050 & 0.0053 & -0.0002 \end{pmatrix}$$

$$\boldsymbol{B}_{1,1} = 10^3 \begin{pmatrix} 0 \\ 0 \\ 0.0026 \\ 0.0100 \end{pmatrix}$$

$$\boldsymbol{A}_{2,1} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -231.3804 & -64.9720 & -4.9178 & -0.3229 \\ 277.9906 & -916.6567 & 12.6582 & 0.6542 \end{pmatrix}$$

$$\boldsymbol{B}_{2,1} = \begin{pmatrix} 0 \\ 0 \\ -36.9511 \\ -139.6544 \end{pmatrix}$$

$$\boldsymbol{A}_{3,1} = 10^3 \begin{pmatrix} 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 \\ -0.2314 & -0.0270 & -0.0038 & -0.0002 \\ 0.2780 & -1.0577 & 0.0086 & 0.0002 \end{pmatrix}$$

$$\boldsymbol{B}_{3,1} = 10^3 \begin{pmatrix} 0 \\ 0 \\ -0.0176 \\ -0.0664 \end{pmatrix}$$

$$\boldsymbol{A}_{1,2} = 10^3 \begin{pmatrix} 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 \\ -0.2314 & -0.0029 & -0.0029 & -0.0001 \\ 0.2780 & 1.2608 & 0.0053 & -0.0002 \end{pmatrix}$$

$$\boldsymbol{B}_{1,2} = 10^3 \begin{pmatrix} 0 \\ 0 \\ 0.0026 \\ 0.0100 \end{pmatrix}$$

$$\boldsymbol{A}_{2,2} = 10^3 \begin{pmatrix} 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 \\ -0.2314 & -0.0805 & -0.0049 & -0.0003 \\ 0.2780 & 1.5491 & 0.0127 & 0.0007 \end{pmatrix}$$

$$\boldsymbol{B}_{2,2} = 10^3 \begin{pmatrix} 0 \\ 0 \\ -0.0370 \\ -0.1397 \end{pmatrix}$$

$$\boldsymbol{A}_{3,2} = 10^3 \begin{pmatrix} 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0 & 0.0010 \\ -0.2314 & -0.0425 & -0.0038 & -0.0002 \\ 0.2780 & 1.4080 & 0.0086 & 0.0002 \end{pmatrix}$$

$$\boldsymbol{B}_{3,2} = 10^3 \begin{pmatrix} 0 \\ 0 \\ -0.0176 \\ -0.0664 \end{pmatrix}$$

B. Observer Design

Reference 1 applied TP model 1 to controller design. The numerical results are detailed in Ref. 1. This section is one of the novelties of this Note and derives an observer for the prototypical aeroelastic wing section using the above obtained TP model 2. The observer design is done under the PDC framework like the controller design discussed in Ref. 1. We use the observer structure:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}(\boldsymbol{p}(t))\hat{\boldsymbol{x}}(t) + \boldsymbol{B}(\boldsymbol{p}(t))\boldsymbol{u}(t) + \boldsymbol{K}(\boldsymbol{p}(t))(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))$$

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{C}(\boldsymbol{p}(t))\hat{\boldsymbol{x}}(t)$$

which in TP model form is

$$\dot{\hat{\boldsymbol{x}}}(t) = A \underset{n}{\otimes} \boldsymbol{w}(p_n(t))\hat{\boldsymbol{x}}(t) + B \underset{n}{\otimes} \boldsymbol{w}_n(p_n(t))\boldsymbol{u}(t) + K \underset{n}{\otimes} \boldsymbol{w}(p_n(t))(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))$$

$$\hat{\boldsymbol{y}}(t) = C \underset{n}{\otimes} \boldsymbol{w}(p_n(t))\hat{\boldsymbol{x}}(t) \tag{9}$$

At this point, we should emphasize that the vector $\boldsymbol{p}(t)$ does not contain values from the estimated state-vector $\hat{\boldsymbol{x}}(t)$ in our case, because $p_1(t)$ equals U and $p_2(t)$ equals the pitch angle $(x_2(t))$. These variables are observable. We estimate only state values $x_3(t)$ and $x_4(t)$. Consequently, the goal in the present case is to determine gains in tensor K for (9). For this goal, the following LMI theorem can be readily derived from Theorem 1 of Ref. 1:

Theorem 1 (Globally and asymptotically stable observer): To ensure $\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \to 0$ as $t \to \infty$, in the observer strategy (9), find $\boldsymbol{P} > 0$ and $\boldsymbol{N}_r$ satisfying the following LMIs:

$$-\boldsymbol{A}_r^T \boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}_r + \boldsymbol{C}_r^T \boldsymbol{N}_r^T + \boldsymbol{N}_r \boldsymbol{C}_r > 0 \tag{10}$$

for all r and

$$-\boldsymbol{A}_r^T \boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}_r - \boldsymbol{A}_s^T \boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}_s + \boldsymbol{C}_r^T \boldsymbol{N}_s^T + \boldsymbol{N}_s \boldsymbol{C}_r + \boldsymbol{C}_s^T \boldsymbol{N}_r^T + \boldsymbol{N}_r \boldsymbol{C}_s > 0 \tag{11}$$

for $r < s \le R$, except the pairs (r, s) such that $w_r(\boldsymbol{p}(t))w_s(\boldsymbol{p}(t)) = 0, \forall \boldsymbol{p}(t)$.

Since the above equations are LMIs, with respect to variables $\boldsymbol{P}$ and $\boldsymbol{N}_r$, we can find a positive definite matrix $\boldsymbol{P}$ and matrix $\boldsymbol{N}_r$ or determine that no such matrices exist. The observer gains can then be obtained as

$$\boldsymbol{K}_r = \boldsymbol{P}^{-1}\boldsymbol{N}_r \tag{12}$$

Finally, by the help of $r = \text{ordering}\,(i_1, i_2, \ldots, i_N)$ defined in Ref. 1 (index transformation) one can define $\boldsymbol{K}_{i_1,i_2,\ldots,i_N}$ from $\boldsymbol{K}_r$ obtained in (12) and store into tensor K of (9).

We now apply Theorem 1 to the TP model 2 of the aeroelastic wing section. We define matrix $\boldsymbol{C}$ for all r from

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$$

which in the present case is

$$\boldsymbol{C}_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The LMIs of Theorem 1, applied to the TP model 2, are feasible. Thus, Eq. (12) yields six observer feedbacks, $\boldsymbol{K}_{i=1,\ldots,3,\,j=1,2}$.

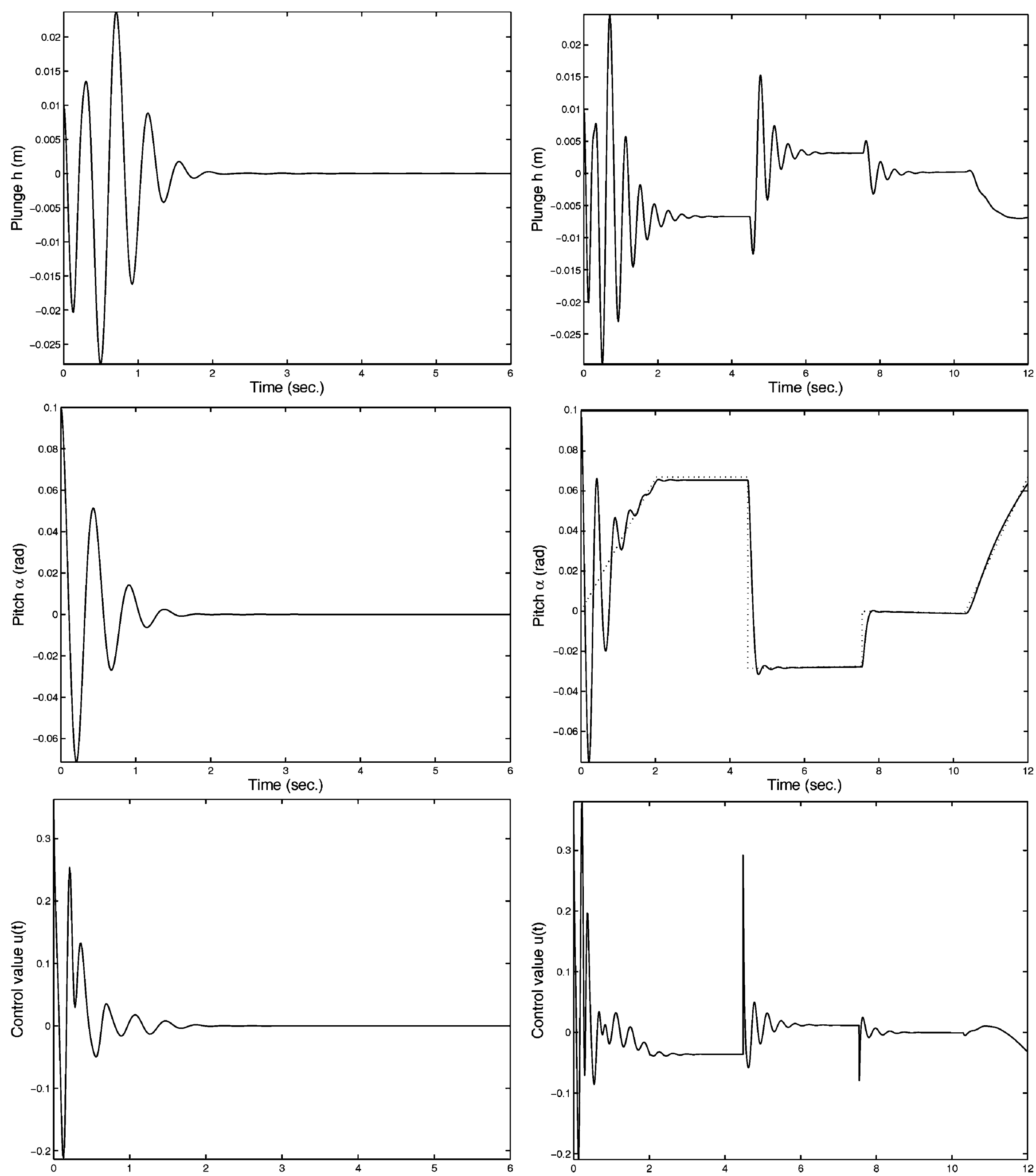

Fig. 4 Asymptotic stabilization by the derived controller (left panel) and trajectory (dashed line) control of the pitch angle (right panel) for $U = 20$ m/s and $a = -0.4$.

In conclusion, the state values $x_3(t)$ and $x_4(t)$ are estimated by (10) as

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}(\boldsymbol{p}(t))\hat{\boldsymbol{x}}(t) + \boldsymbol{B}(\boldsymbol{p}(t))u(t) + \left(\sum_{i=1}^{3}\sum_{j=1}^{2} w_{1,i}(U)w_{2,j}(\alpha)\boldsymbol{K}_{i,j}\right)(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))$$

where

$$\boldsymbol{y}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \hat{\boldsymbol{y}}(t) = \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{p}(t) = \begin{pmatrix} U \\ \alpha \end{pmatrix}$$

At this point it is important to emphasize that if we apply TP model 1, then the resulted control system is not capable of stabilizing the present aeroelastic system. The design process does not need analytic interaction and takes a few minutes on a regular computer. We can easily guarantee various design specifications beyond stability by selecting proper LMI conditions.

V. Simulation Results

To be comparable to other published results, the numerical examples are performed with $a = -0.4$ and with free stream velocity $U = 20$ m/s, a velocity that exceeds the linear flutter velocity, $U = 15.5$ m/s, and for initial conditions $h = 0.01$ m and $\alpha = 0.1$ rad. The initial observer state is $\hat{\boldsymbol{x}}(t) = (0\ \ 0\ \ 0\ \ 0)$. The left panel of Fig. 4 shows that the system is asymptotically stabilized. We can see that the stabilization time is a bit longer with the observer, than without the observer, as given in Ref. 1. We can also observe that the convergence of the pitch angle is much more smooth without the observer in Ref. 1. If we compare the result derived here to the control result of the feedback linearization (see Fig. 7 of Ref. 1), we can conclude that the output feedback control derived here is faster. The right panel of Fig. 4 shows how the system is capable of tracking the trajectory command of the pitch angle. The command trajectory has no practical aspect here; it is only a theoretical command to show the response of the controller system for step and ramp functions. One can see that the pitch angle converges to the command signal in 1–2 s.

VI. Conclusions

This Note proposes a strategy of control by output feedback to the prototypical aeroelastic wing section. The control solution includes a recently published state-feedback controller and an observer, derived here, to estimate the practically unmeasurable state values form the output. Both the controller and the observer are designed with the help of the TP model transformation and the feasibility test of LMIs under the PDC framework. To relax the observer design, this Note proposes extra constraints on the coefficient functions of the TP model and extends the TP model transformation with an extra step capable of satisfying this constraint. The Note also presents numerical simulations to validate the derived controller and observer.

References

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[2] Yam, Y., Baranyi, P., and Yang, C. T.,"Reduction of Fuzzy Rule Base via Singular Value Decomposition," *IEEE Transactions on Fuzzy Systems*, Vol. 7, No. 2, 1999, pp. 120–132.